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A minimum variance unbiased estimator of finite population variance using auxiliary information

Archana Panigrahi [1], Priyaranjan Dash[2], Gopabandhu Mishra[3]

Abstract

A class of estimators of finite population variance (S_y^2) using auxiliary information has been proposed under simple random sampling without replacement (SRSWOR) scheme. An attempt has been made to derive the minimum variance unbiased estimator of finite population variance from the proposed class of unbiased estimators. The efficiency of the class of estimators under optimality is compared with the usual unbiased estimator ($t_{V0} = s_y^2$), ratio type estimator *(t_{VR})*, product type estimator (t_{VP}), regression type estimator (t_{Vlr}), exponential ratio type estimator ($\mathrm{t_{VER}}$) , exponential product type estimator (t_{VEP}), and ratio-in-regression estimator ($\mathrm{t_s}$), both theoretically and empirically under general conditions and under bivariate normality. The proposed class of estimator performs better than these estimators under certain realistic conditions. The proposed class of estimators is generalized for the case of multi-auxiliary variables.

Key words: simple random sampling without replacement (SRSWOR), unbiased estimator, auxiliary variable, population variance, efficiency.

1. Introduction

Estimation of finite population variance draws attention of many researchers due to its wide spread applications in various fields like portfolio analysis, asset evaluation, stock market analysis in finance; quality control problems, traffic controls, opinion polls, biostatistics, logistics analysis and many more. So, researchers have been paying specific attention towards estimating the finite population variance $\left(S_y^2\right)$ with a greater precision for many decades. Consider a finite population of N $(< \infty)$ units and Y_i being

[1] Ravenshaw University, Cuttack, Odisha, India. E-mail: archanapanigrahi10@gmail.com. ORCID: https://orcid.org/ 0009-0006-3349-9524.
[2] Utkal University, Vanivihar, Bhubaneswar, Odisha, India. E-mail: prdashjsp@gmail.com. ORCID: https://orcid.org/ 0000-0001-6561-4095.
[3] Utkal University, Vanivihar, Bhubaneswar, Odisha, India. E-mail: gmishrauu@gmail.com. ORCID: https://orcid.org/ 0000-0003-3959-2333.

the value of the study variable y for the i-th unit of the population ($i = 1(1)N$). We denote

$$S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \overline{Y})^2 \tag{1}$$

as the finite population variance. In order to estimate S_y^2, we draw a sample of size n from the population of size N using simple random sampling without replacement (SRSWOR) scheme. Suppose y_i is the value of the study variable y for the i-th unit of the sample ($i = 1(1)n$).

To start with a very simple popular estimator of finite population variance, we consider the sample variance

$$t_{V0} = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \overline{y})^2 = s_y^2, \tag{2}$$

which is an unbiased estimator of S_y^2 and its variance is given by

$$V(t_{V0}) = \theta S_y^4[\beta_{04} - 1], \tag{3}$$

where $\theta = \frac{1}{n} - \frac{1}{N}$, $\beta_{04} = \frac{\mu_{04}}{{\mu_{02}}^2}$, $\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \overline{X})^r(Y_i - \overline{Y})^s$

and μ_{rs} is called the (r,s)-th order bivariate central moment of (x,y) for the population.

In order to improve the present estimator by exploiting the information on an auxiliary variable x (if available), Isaki (1983) proposed a ratio estimator for finite population variance using advance knowledge of population variance of auxiliary variable S_x^2 as

$$t_{VR} = s_y^2\frac{S_x^2}{s_x^2} \tag{4}$$

The estimator t_{VR} is a biased estimator of S_y^2 and its MSE up to $o(n^{-1})$ is given by

$$MSE(t_{VR}) = \theta S_y^2[\beta_{04} + \beta_{40} - 2\beta_{22}], where\ \beta_{40} = \frac{\mu_{40}}{{\mu_{20}}^2}, \beta_{22} = \frac{\mu_{22}}{\mu_{20}\,\mu_{02}}. \tag{5}$$

It may be seen that t_{VR} is more efficient than t_{V0} if

$$\S > \frac{1}{2}\frac{C.V.(s_x^2)}{C.V.(s_y^2)}, \tag{6}$$

where § denotes the correlation coefficient between s_x^2 and s_y^2. Further, $C.V.(s_x^2)$ and $C.V.(s_y^2)$ denote the coefficient of variation of s_x^2 and s_y^2 respectively. Thus, t_{VR} is conditionally more efficient than the usual variance estimator t_{V0}, which does not use any auxiliary information.

Following Isaki (1983), a product estimator for S_y^2 is suggested as

$$t_{VP} = s_y^2 \frac{s_x^2}{S_x^2}. \tag{7}$$

It is also a biased estimator of S_y^2 and its MSE up to $o(n^{-1})$ is given by

$$MSE(t_{VP}) = \theta S_y^4 [\beta_{04} + \beta_{40} + 2\beta_{22} - 4] \tag{8}$$

Isaki (1983) also suggested a regression type estimator for estimating S_y^2, which is given by

$$t_{VLR} = s_y^2 + \hat{B}(S_x^2 - s_x^2) \tag{9}$$

where $B = \frac{Cov(s_x^2, s_x^2)}{V(s_x^2)}$ and $\hat{B}$ is the estimated value of B.

This estimator t_{VLR} is also a biased estimator of S_y^2 and its MSE up to $o(n^{-1})$ is given by

$$MSE(t_{VLR}) = \theta S_y^4 [\beta_{04} - 1]\,[1 - §^2] \tag{10}$$

Singh et al. (2009) suggested an exponential ratio-type and an exponential product-type estimator for S_y^2, which are given by

$$t_{VER} = s_y^2 \exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) \tag{11}$$

and

$$t_{VEP} = s_y^2 \exp\left(\frac{S_x^2 + s_x^2}{S_x^2 - s_x^2}\right). \tag{12}$$

respectively. Both these estimators are biased estimators of S_y^2 and their MSE up to $o(n^{-1})$ are given by

$$MSE(t_{VER}) = \theta S_y^4 \left[\beta_{04} + \frac{1}{4}\beta_{40} - \beta_{22} - \frac{1}{4}\right] \tag{13}$$

and

$$MSE(t_{VEP}) = \theta S_y^4 \left[\beta_{04} + \frac{1}{4}\beta_{40} + \beta_{22} - \frac{9}{4}\right] \tag{14}$$

respectively.

Swain (2015) suggested a class of ratio-in-regression estimators for estimating the finite population variance $\left(S_y^2\right)$ as

$$t_s = ws_y^2 + (1-w)s_y^2\left(\frac{S_x^2}{s_x^2}\right) \tag{15}$$

The value of w for which the above class of estimators attains a minimum variance is given by

$$w_{opt} = \beta_{40} - \beta_{22}.$$

Using this value of w in t_s, we get the optimum estimator (estimator attaining a minimum variance) for this class as

$$t_{s(opt)} = (\beta_{40} - \beta_{22})s_y^2 + (1 - \beta_{40} + \beta_{22})s_y^2\left(\frac{S_x^2}{s_x^2}\right). \tag{16}$$

The mean square error of this estimator is given by

$$MSE\left(t_{s(opt)}\right) = \theta S_y^4\left[(\beta_{04} - 1) - \frac{(\beta_{22} - 1)^2}{\beta_{40} - 1}\right]. \tag{17}$$

For a brief review of literature about the variance estimation, one can go through Evans (1951), Wakimoto (1971), Liu (1974), Das and Tripathi (1978), Srivastava and Jhajj (1980), Wu (1982), Upadhyaya and Singh (1983), Singh (1986), Singh et al. (1988), Prasad and Singh (1990,1992), Swain and Mishra (1992, 1994a,b), Singh and Biradar (1994), Cebrian and Garcia (1997), Kadilar and Cingi (2007), Shabbir and Gupta (2007), Dubey and Sharma (2008), Yadav et al. (2013), Singh and Solanki (2013a,b), Singh et al. (2014), Dash and Sunani (2019) and many more.

2. Proposed class of unbiased estimators

A class of estimators for estimating finite population variance (S_y^2) in the presence of auxiliary information is proposed by taking the linear combination of the sample variance of study variable (s_y^2) and the sample variance of auxiliary variable (s_x^2). The proposed class of estimators of S_y^2 is given by

$$t_{VM} = \lambda_0 s_y^2 + \lambda_1 s_x^2, \tag{18}$$

where λ_0 and λ_1 are two suitable constants or statistics. For different values of λ_0 and $\lambda_1$one can get different estimators. The following table presents some particular estimators deduced from the proposed class of estimator for different suitably chosen values of λ_0 and λ_1 .

Table 1: Some Deduced Estimators

λ_0	λ_1	Deduced estimators
1	0	$t_{V0} = s_y^2$ [Usual unbiased estimator]
$\dfrac{S_x^2}{s_x^2}$	0	$t_{VR} = s_y^2 \dfrac{S_x^2}{s_x^2}$ [Isaki (1983)]
$\dfrac{s_x^2}{S_x^2}$	0	$t_{VP} = s_y^2 \dfrac{s_x^2}{S_x^2}$
1	$\dfrac{(S_x^2 - s_x^2)}{s_x^2}$	$t_{Vlr} = s_y^2 + \widehat{B}(S_x^2 - s_x^2)$ [Isaki (1983)]
$\exp\left[\dfrac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$	0	$t_{VER} = s_y^2 \exp\left[\dfrac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right]$, [Singh et al. (2009)]
$\exp\left[\dfrac{s_x^2 + S_x^2}{s_x^2 - S_x^2}\right]$	0	$t_{VEP} = s_y^2 \exp\left[\dfrac{S_x^2 + s_x^2}{S_x^2 - s_x^2}\right]$ [Singh et al. (2009)]
$\beta_{40} - \beta_{22}$	$(1 - \lambda_0)\dfrac{s_y^2}{s_x^2}\dfrac{S_x^2}{s_x^2}$	$t_{s(opt)} = \lambda_0 s_y^2 + (1 - \lambda_0)\, t_{VR}$, [Swain, 2015]

3. Variance of the proposed class of estimators

To construct a class of unbiased estimators, we have to choose the constants λ_0 and λ_1 such that the class of estimators in (18) is unbiased for S_y^2 and we shall attempt to identify minimum variance unbiased estimator in this class. The condition for t_{VM} to be unbiased for S_y^2 is given by

$$(\lambda_0 - 1)S_y^2 + \lambda_1 S_x^2 = 0 \tag{19}$$

The variance of the estimator t_{VM} is

$$V(t_{VM}) = \lambda' S_2 \lambda, \tag{20}$$

where

$$\lambda = \begin{bmatrix}\lambda_0\\ \lambda_1\end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} S_{00} & S_{01}\\ S_{10} & S_{11}\end{bmatrix},$$

$$S_{00} = V(s_y^2) = \theta S_y^4[\beta_2(y) - 1], \quad S_{11} = V(s_x^2) = \theta S_x^4[\beta_2(x) - 1]$$

$$S_{10} = S_{01} = \mathrm{Cov}(s_x^2, s_y^2) = \theta S_x^2 S_y^2[\beta_{22}(x,y) - 1].$$

The optimum value of λ satisfying the unbiasedness condition (19) and leading to a minimum variance of t_{VM} is

$$\lambda_{opt} = S_y^2 \frac{S_2^{-1} Q_2}{Q_2' S_2^{-1} Q_2} = \lambda^*, \tag{21}$$

where S_2^{-1} is inverse of matrix S_2, $Q_2 = \begin{bmatrix} S_y^2 \\ S_x^2 \end{bmatrix}$, Q_2' is the transpose of Q_2.

Hence, after expansion, we get

$$\lambda^* = \frac{S_y^2\begin{bmatrix} S_y^2 S_{11} - S_x^2 S_{10} \\ -S_y^2 S_{10} + S_x^2 S_{00}\end{bmatrix}}{S_y^4 S_{11} - 2S_y^2 S_x^2 S_{10} + S_x^4 S_{00}} \tag{22}$$

From (22) we get optimum values of λ_0 and λ_1, which are denoted by λ_0^* and λ_1^* where

$$\lambda_0^* = S_y^2 \frac{(S_y^2 S_{11} - S_x^2 S_{10})}{S_y^4 S_{11} - 2S_y^2 S_x^2 S_{10} + S_x^4 S_{00}} = \frac{\beta_{40} - \beta_{22}}{\beta_{40} - 2\beta_{22} + \beta_{04}}, \tag{23}$$

$$\lambda_1^* = S_y^2 \frac{S_x^2 S_{00} - S_y^2 S_{10}}{S_y^4 S_{11} - 2S_y^2 S_x^2 S_{10} + S_x^4 S_{00}} = \frac{S_y^2}{S_x^2} \frac{\beta_{04} - \beta_{22}}{\beta_{40} - 2\beta_{22} + \beta_{04}} \tag{24}$$

are called optimality conditions.

The minimum variance unbiased estimator of population variance is given by

$$t_{VM(o)} = \lambda_0^* s_y^2 + \lambda_1^* s_x^2, \tag{25}$$

Using the values of λ_0^* and λ_1^* in equation (20), the minimum(optimum) variance of t_{VM} is given by

$$V\left(t_{VM(o)}\right)_{opt} = \frac{S_y^4}{Q_2' \, S_2^{-1} \, Q_2} = S_y^4 \frac{S_{00} S_{11} - S_{10}^2}{S_y^4 S_{11} - 2S_y^2 S_x^2 S_{10} + S_x^4 S_{00}} \tag{26}$$

$$= \theta S_y^4 \frac{[\beta_{40} - 1][\beta_{04} - 1]\,[1 - \S^2]}{\beta_{40} - 2\beta_{22} + \beta_{04}}. \tag{27}$$

4. Comparison of efficiency

The members of the suggested class of estimators t_{VM} are unbiased estimators of the finite population variance S_y^2. Hence, this class of estimators t_{VM} in its optimal case is compared with several other estimators in terms of efficiency only. The comparison of the efficiency of suggested class of estimators t_{VM} under optimality with other estimators are made under the following two conditions:

(a) under general condition and

(b) under bivariate normality condition.

I. The suggested class of estimators t_{VM} under optimality $\left(t_{VM(o)}\right)$ is always more efficient than the usual unbiased estimator t_{V0} .

II. The class of estimators t_{VM} under optimality $\left(t_{VM(o)}\right)$ is more efficient than t_{VR} if

Case (a): $[\beta_{40} - 2\beta_{22} + \beta_{04}]^2 > [\beta_{40} - 1][\beta_{04} - 1]\left[1 - \S^2\right]$ (28)

Case (b): $4(1 - \rho^2)^2 > [1 - \S^2],$ (29)

where, ρ is the correlation coefficient between the study variable y and the auxiliary variable x.

III. The class of estimators t_{VM} under optimality $\left(t_{VM(o)}\right)$ is more efficient than t_{VP} if

Case(a): $[\beta_{40} - 2\beta_{22} + \beta_{04}]^2 - 4[\beta_{22} - 1]^2$
$> [\beta_{40} - 1][\beta_{04} - 1][1 - \S^2]$ (30)

Case(b): $4(1 - 2\rho^2) > [1 - \S^2]$ (31)

IV. The class of estimators t_{VM} under optimality $\left(t_{VM(o)}\right)$ is more efficient than t_{Vlr} if

Case(a): $$[\beta_{04}+1] > 2\beta_{22} \tag{32}$$

Case(b): $$\rho^2 < \frac{1}{2} \tag{33}$$

V. The class of estimators t_{VM} under optimality $\left(t_{VM(o)}\right)$ is more efficient than t_{VER} if

Case(a): $$[\beta_{40}-2\beta_{22}+\beta_{04}]\left[\beta_{40}-2\beta_{22}+\beta_{04}-\frac{1}{4}\right] > [\beta_{40}-1][\beta_{04}-1][1-\S^2] \tag{34}$$

Case(b): $$(1-\rho^2)(15-16\rho^2) > 4[1-\S^2] \tag{35}$$

VI. The class of estimator t_{VM} under optimality $\left(t_{VM(o)}\right)$ is more efficient than t_{VEP} if

Case(a): $$[\beta_{40}-2\beta_{22}+\beta_{04}]\left[\frac{1}{4}\beta_{40}-2\beta_{22}+\beta_{04}-\frac{9}{4}\right] > [\beta_{40}-1][\beta_{04}-1][1-\S^2] \tag{36}$$

Case(b): $$(1-\rho^2)(15-16\rho^2) > 4[1-\S^2] \tag{37}$$

VII. The class of estimator t_{VM} under optimality $\left(t_{VM(o)}\right)$ is more efficient than $t_{s(opt)}$ if

Case(a): $$(\beta_{04}-1)(\beta_{40}-1)-(\beta_{22}-1)^2[\beta_{40}-2\beta_{22}+\beta_{04}] > [\beta_{40}-1]^2[\beta_{04}-1][1-\S^2] \tag{38}$$

Case(b): $$2(1-\rho^4)(1-\rho^2) > [1-\S^2] \tag{39}$$

5. Empirical Study

To study the performance of the proposed class of estimators numerically, we consider 16 populations collected from different literature as described in Table 2 (8 populations having positive correlation between x and y) and in Table 3 (8 populations having negative correlation between x and y). The source of populations, description of the nature of x and y, N, n, ρ, β_{40} and β_{04} for these populations are presented in these tables. We have considered the usual unbiased estimator ($t_{V0} = s_y^2$), ratio type estimator (t_{VR}) due to Isaki (1983), product type estimator (t_{VP}), regression type estimator (t_{Vlr}) due to Isaki (1983), exponential ratio type estimator (t_{VER}) due to Singh et al. (2009), exponential product type estimator (t_{VEP}) due to Singh et al. (2009), Swain (2015) ratio-in-regression estimator ($t_{s(opt)}$) in order to compare with the proposed class of estimators under optimality, i.e. the minimum variance unbiased estimator $\left(t_{VM(o)}\right)$ of the proposed class. Assuming simple random sampling without replacement, the relative efficiency of different estimators with respect to t_{V0} is compiled in Table 3 (for the cases in which the correlation coefficient is positive) and Table 4 (for the cases in which the correlation coefficient is negative).

Table 2: Description of the Populations ($\rho > 0$)

Pop^n No.	Source	x	y	N	ρ	β_{40}	β_{04}
1	Jobson (1992), p .674	Percentage of White People in Population	Total Mortality Rate	80	0.18	1.93	3.261
2	Draper & Smith (1966), p. 366	Amount of Tricalcium Silicate	Amount of Tricalcium Aluminate	13	0.23	1.677	3.075
3	Black (2009), p.517	Job Satisfaction	Advancement Opportunities	19	0.26	2.737	1.889
4	Stevens (2009), p. 81	Knowledge	Instructor Evaluation	32	0.28	3.099	2.511
5	Draper & Smith (1966), p.352	Pounds of Crude Glycerin Made	Pounds of Steam used monthly	25	0.31	3.053	2.278
6	Murthy (1967), p. 91	Holding Size (in acres)	Cultivated Area (in acres)	36	0.37	17.47	3.183
7	Cochran (1953), p. 113	Size of large US Cities in 1920	Size of large US Cities in 1930	49	0.4	7.504	38.91
8	Black (2009), p. 517	Relationship with Supervisor	Advancement Opportunities	19	0.4	2.654	1.889

Table 3: Description of the Populations ($\rho < 0$)

Pop^n No.	Source	x	y	N	ρ	β_{40}	β_{04}
1	Black (2009), p. 511	Time Period	Total No. of Savings Organization	13	-0.17	1.786	1.602
2	Maddala (1988), p. 197	Year	Prices Received by Farmer for Food	20	-0.21	1.794	2.81
3	Draper & Smith (1966), p. 352	Number of Start-ups	Average Atmospheric Temperature	25	-0.24	4.075	1.532
4	Härdle & Hlávka (2007), p.335	Marks for Car Safety	Marks for Car Economy	24	-0.27	3.346	2.684
5	Maddala (1988), p. 197	Year	Food Prod. per Capita	20	-0.34	1.794	2.453
6	Black (2009), p. 567	Nuclear Electricity Production (in billion kwh)	Dry Gas Prod. (in trillions of cubic feet)	26	-0.4	1.579	3.012
7	Black (2009), p.567	Dry Gas Prod. (in trillions of cubic feet)	Fuel Rate for Automobile (in miles/ gallon)	26	-0.42	1.599	3.012
8	Härdle & Hlávka (2007), p. 339	Expenditure on Meat	Expenditure on Wine	12	-0.44	2.056	1.837

Table 4: PRE of Estimators with respect **to t_{V0}** (when $\rho > 0$)

Pop^n No.	N	n	Estimators					
			t_{V0}	t_{VR}	t_{VER}	t_{Vlr}	$t_{s(opt)}$	$t_{VM(o)}$
1	80	32	100	65.403	86.088	100.84	124.56	374.65
2	13	6	100	57.714	77.839	114.46	251.23	607.36
3	19	8	100	28.694	57.014	103.74	114.56	184.99
4	32	13	100	37.636	67.504	101.32	126.19	193.88
5	25	10	100	53.811	97.319	109.54	110.44	126.71
6	36	15	100	11.286	32.855	100.33	105.30	117.8
7	49	20	100	85.608	96.046	100	264.98	680.84
8	19	8	100	28.613	56.103	105.73	137.15	198.64

Table 5: PRE of Estimators with respect **to t_{V0}** (when $\rho < 0$)

Pop^n No.	N	n	Estimators					
			t_{V0}	t_{VR}	t_{VER}	t_{Vlr}	$t_{s(opt)}$	$t_{VM(o)}$
1	13	6	100	33.164	59.471	110.71	122.31	135.27
2	20	8	100	131.08	129.57	135.18	269.34	651.64
3	25	10	100	14.945	41.679	100.04	102.23	118.96
4	24	10	100	47.729	83.372	101.61	149.31	196.28
5	20	8	100	56.408	80.016	102.4	165.42	247.35
6	26	11	100	63.243	82.057	108.09	197.85	373.49
7	26	11	100	72.962	90.02	100.45	254.66	413.27
8	12	5	100	42.041	72.777	100.28	141.64	170.44

6. Extension to the case of multi-auxiliary variables

The proposed class of unbiased estimators can be extended to the case of multi-auxiliary variables when the information on population variances $s_{x_1}^2, s_{x_2}^2, \dots, s_{x_k}^2$ for $k -$ auxiliary variables is available. The class of unbiased-estimators of S_y^2 is

$$T = \lambda_0 s_y^2 + \lambda_1 s_{x_1}^2 + \cdots + \lambda_k s_{x_k}^2, \tag{40}$$

where the $(k+1)$ constants λ_i $(i = 0,1,2,\cdots,k)$ are chosen so that T will be unbiased for S_y^2. So, we have

$$E(T) = S_y^2 \quad \text{i.e. } \lambda' \, Q_{k+1} = S_y^2 \tag{41}$$

and the variance of the class of estimators T is

$$V(T) = \lambda' S_{k+1} \lambda, \tag{42}$$

where $\lambda' = [\lambda_0, \lambda_1, \cdots, \lambda_k]$ is a matrix of order $1 \times (k+1)$ and $Q_{k+1}{}' = [S_y^2, S_{x_1}^2, \cdots, S_{x_k}^2]$ is a matrix of order $1 \times (k+1)$; $S_{k+1} = [S_{ij}]$ is the dispersion matrix of order $(k+1)$, such that $S_{00} = Var(s_y^2)$, $S_{0j} = Cov\left(s_y^2, s_{x_j}^2\right) = S_{j0}$, $S_{ij} = Cov\left(s_{x_i}^2, s_{x_j}^2\right) = S_{ji}$ $j = 1, \cdots, k, i = 1,2, \cdots, k$.

The value of λ which minimises (41) subject to the unbiasedness condition (40) is found to be

$$\lambda_{opt} = \mathrm{S}_{\mathrm{y}}^2 \frac{S_{k+1}^{-1} \mathrm{Q}_{k+1}}{\mathrm{Q}_{k+1}{}' S_{k+1}^{-1} \mathrm{Q}_{k+1}} = \lambda^*(say), \tag{43}$$

where S_{k+1}^{-1} is inverse of the matrix S_{k+1} and $\lambda^{*\prime} = [\lambda_0^*, \lambda_1^*, \cdots, \lambda_k^*]$ are the optimum values of λ'. From the above conditions, we get the minimum variance unbiased estimator of this class as

$$\mathrm{T}_{opt} = \lambda_0^* \mathrm{s}_{\mathrm{y}}^2 + \lambda_2^* \mathrm{s}_{\mathrm{x}_1}^2 + \cdots + \lambda_k^* \mathrm{s}_{\mathrm{x}_\mathrm{k}}^2. \tag{44}$$

The optimum variance of T_{opt} is given by

$$\mathrm{V}(\mathrm{T}_{opt}) = \frac{\mathrm{S}_{\mathrm{y}}^4}{\mathrm{Q}_{k+1}{}' S_{k+1}^{-1} \mathrm{Q}_{k+1}} \tag{45}$$

7. Conclusion

1) In Table 4, for populations 1 to 8 the percent relative efficiency of the minimum variance unbiased estimator $\left(\mathrm{t}_{\mathrm{VM(o)}}\right)$ of the proposed class is maximal.
2) In Table 5, for populations 1 to 8 the percent relative efficiency of the minimum variance unbiased estimator $\left(\mathrm{t}_{\mathrm{VM(o)}}\right)$ of the proposed class is maximal.

When the study variable and auxiliary variable are positively correlated, the proposed estimator is more efficient than the ratio type estimator $(\mathrm{t}_{\mathrm{VR}})$, exponential ratio type estimator $(\mathrm{t}_{\mathrm{VER}})$, regression type estimator $(\mathrm{t}_{\mathrm{Vlr}})$ and ratio-in-regression estimator $(\mathrm{t}_{\mathrm{s(opt)}})$. We may write the efficiency (E) in descending order

$$\mathrm{E}\left(\mathrm{t}_{\mathrm{VM(o)}}\right) \geq \mathrm{E}\left(\mathrm{t}_{\mathrm{s(opt)}}\right) \geq \mathrm{E}(\mathrm{t}_{\mathrm{Vlr}}) \geq \mathrm{E}(\mathrm{t}_{\mathrm{VER}}) \geq \mathrm{E}(\mathrm{t}_{\mathrm{VR}}).$$

When the study variable and auxiliary variable are negatively correlated, the proposed estimator is more efficient than the product type estimator $(\mathrm{t}_{\mathrm{VP}})$, exponential product type estimator $(\mathrm{t}_{\mathrm{VEP}})$, regression type estimator $(\mathrm{t}_{\mathrm{Vlr}})$ and ratio-in-regression estimator $(\mathrm{t}_{\mathrm{s(opt)}})$. We may write the efficiency (E) in descending order

$$\mathrm{E}\left(\mathrm{t}_{\mathrm{VM(o)}}\right) \geq \mathrm{E}\left(\mathrm{t}_{\mathrm{s(opt)}}\right) \geq \mathrm{E}(\mathrm{t}_{\mathrm{Vlr}}) \geq \mathrm{E}(\mathrm{t}_{\mathrm{VEP}}) \geq \mathrm{E}(\mathrm{t}_{\mathrm{VP}}).$$

References

Black, K., (2009). Business Statistics for Contemporary Decision Making, 6th Edition. *John Wiley and Sons*, Inc., New York, USA.

Cebrian, A. A., Garcia, M. R., (1997). Variance estimation using auxiliary information: An almost unbiased multivariate ratio estimator. *Metrika*, Vol. 45, pp. 171–178.

Cochran, W. G., (1977). Contribution to Theory of Sampling, 3rd Edition. *Wiley Eastern Limited*, India.

Das, A. K., Tripathi, T. P., (1978). Use of auxiliary information in estimating the finite population variance. *Sankhya C*, Vol. 40, pp. 139–148.

Dash, P. R., Sunani, K., (2019). Estimation of Finite Population Variance Using A Family of Exponential Estimators. *International Journal of Research and Analytical Reviews*, Vol. 6(6), pp. 1200–1206.

Draper, N. R., Smith, H., (1966). Applied Regression Analysis, 1st Edition, John Wiley and Sons Inc., New York, USA.

Dubey, V., Sharma, H. K., (2008). On estimating population variance using auxiliary information. *Statist. Trans. New Ser.*, 9(1), pp. 7–18.

Evans, D., (1951). On the variance estimators of SD and Variance. *Jour. Amer. Stat. Assoc.,* 46, pp. 220–224.

Hardle, W., Hlavka, Z., (2007). Multivariate Statistics: Exercise and Solutions. *Springer Science+Business Media*, LLC, New York, USA.

Isaki, C. T., (1983). Variance estimation using auxiliary information. *Jour. Amer. Stat. Assoc.*, Vol. 78(381), pp. 117–123.

Jobson, J. D., (1992). Applied Multivariate Data Analysis Volume II: Categorical and Multivariate Methods. *Springer Science+Business Media*, LLC, New York, USA.

Kadilar, C., Cingi, H., (2007). Improvement in Variance Estimation in Simple Random Sampling. *Communications in Statistics-Theory & Methods*, Vol. 36(11), pp. 2075–2081.

Liu, T. P., (1974). A generalized unbiased estimator for the variance of a finite population. *Sankhya C*, Vol. 36, pp. 23–32.

Maddala, G. S., (1992). Introduction to Econometrics, 2nd Edition. *Macmillan Publishing Company*, New York, USA.

Murthy, M. N., (1967). Sampling Theory and Methods. *Statistical Publishing Society*, Calcutta, India.

Prasad, B., Singh, H. P., (1990). Some improved ratio-type estimators of finite population variance in sample surveys. *Communications in Statistics-Theory & Methods*, Vol. 19(3), pp. 1127–1139.

Prasad, B., Singh, H. P., (1992). Unbiased estimators of finite population variance using auxiliary information in sample surveys. *Communications in Statistics-Theory & Methods*, Vol. 21(5), pp. 1367–1376.

Shabbir, J., Gupta, S., (2007). On improvement in variance estimation using auxiliary information. *Communications in Statistics-Theory & Methods*, Vol. 36(12), pp. 2177–2185.

Singh, H. P., Biradar, R. S., (1994). Estimation of finite population variance using auxiliary information. *Jour. Ind. Soc. Statist. Operas. Res.*, Vol. 15(1–4), pp. 47– 63.

Singh, H. P., Upadhyaya, L. N. and Namjoshi, U. D., (1988). Estimation of finite population variance. *Current Science*, Vol. 57, pp. 1331–1334.

Singh, H. P., (1986). A note on the estimation of variance of sample mean using the knowledge of coefficient of variation in normal population. *Communications in Statistics-Theory & Methods*, Vol. 15(12), pp. 3737–3746.

Singh, H., Singh, A. and Solanki, R., (2014). Estimation of finite population variance using auxiliary information in sample surveys. *Statistica*, Vol. 74(1), pp. 99-116.

Singh, H. P. Solanki, R. S. (2013a). A new procedure for variance estimation in simple random sampling using auxiliary information. *Statistical Papers*, Vol. 54(2), pp. 479–497.

Singh, H. P. Solanki, R. S., (2013b). Improved estimation of finite population variance using auxiliary information. *Communications in Statistics-Theory & Methods*, Vol. 42(15), pp. 2718–2730.

Singh, R., Chauhan, P., Sawan, N. and Smarandache, F., (2009). Improved exponential estimator for population variance using two auxiliary variables. *Octogon Mathematical Magazine*, Vol. 17(2), pp. 667–674.

Srivastava, S. K., Jhajj, H. S., (1980). A class of estimators using auxiliary information for estimating finite population variance. *Sankhya C*, Vol. 42, pp. 87–96.

Stevens, J. P., (2009). Applied Multivariate Statistics for the Social Sciences, 5th Edition. *Routledge*, New York, USA.

Swain, A. K. P. C., Mishra, G., (1994a). Estimation of population variance under unequal probability sampling. *Sankhya B*, Vol. 56, pp. 374–384.

Swain, A. K. P. C., Mishra, G., (1992). Unbiased estimators of finite population variance using auxiliary information. *Metron*, L-n, 3–4, pp. 201–216.

Swain, A. K. P. C., Mishra, G., (1994b). Limiting distribution of the ratio estimator of finite population variance. *Sankhya B*, Vol. 56, pp. 11–17.

Swain, A. K. P. C., (2015). Generalized Estimator of Finite population Variance. *Journal of Statistical Theory and Applications*, Vol. 14(1), pp. 45–51.

Upadhyaya, L. N., Singh H. P., (1983). Use of auxiliary information in the estimation of population variance. *Mathematical Forum*, Vol. VI(2), pp. 33–36.

Wakimoto, K., (1971). Stratified random sampling (I): Estimation of population variance. *Ann. Inst. Statist. Math.*, Vol. 23, pp. 233–252 .

Wu, C. F., (1982). Estimation of variance of the ratio estimator. *Biometrika*, Vol. 69(1), pp. 183–189.

Yadav, R., Upadhyaya, L. N., Singh, H. P., Chatterjee, S., (2013). A generalized family of transformed ratio-product estimators for variance in sample surveys. *Communications in Statistics-Theory & Methods*, Vol. 42(10), pp. 1839–1850.